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Generalised Cornell Potential using Hellmann Theorem: Expectation values for Quark Confinement.

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Abstract

In this research work, we explored non-relativistic quantum mechanics by employing the Nikiforov-Uvarov (NU) method alongside the Schrödinger wave equation, utilizing the generalized Cornell potential model. Through this approach, we derived the bound-state energy spectra and formulated the mass spectra equation. The results were applied to charmonium and bottomonium mesons to provide phenomenological predictions for their mass spectra. Additionally, the Hellmann-Feynman Theorem was utilized to determine the expectation values of the square of the inverse of position $\langle r^{-2} \rangle$, the inverse of position $\langle r^{-1} \rangle$, the square of momentum, $\langle p^2 \rangle$ and the kinetic energy $\langle T \rangle$ for the quantum states of charmonium and bottomonium. The phenomenological predicted results were compared with experimental data, yielding a percentage error of approximately 0.99% for charmonium and 0.44% for bottomonium. The expectation values showed similar characteristics between the charmonium and bottomonium mesons for the same quantum states, with the charmonium meson leading only in the expectation values for, but lagging behind the bottomonium meson in the expectation values for $\langle r^{-2} \rangle$, $\langle r^{-1} \rangle$, and $\langle p^2 \rangle$.

Keywords: *Generalised Cornell potential; Exp*ectation Values; Nikiforov–Uvarov method; inverse position; kinetic energy.

1. Introduction

The study of quark confinement is an essential aspect of quantum chromodynamics (QCD), which is used in describing the phenomenon where quarks exist as bound states; that is, they are perpetually bound within hadrons and have never been observed in experiments to exist as free particles (Greiner and Schramm, 2009). This phenomenon is known to be a direct result of the strong nuclear force, facilitated by gluons, which becomes increasingly stronger as quarks separate, eventually leading to the formation of hadrons (Zalewski, 1998). The Cornell potential is a widely used model that effectively describes quark-antiquark interactions through a combination of Coulomb and linear scalar potentials (Eichten et al., 1978). This potential has been instrumental in understanding the quarkonium mass spectra and its bound states.

However, most studies suggest that a generalized Cornell potential, with additional terms, can provide a more accurate representation of quark confinement (Vega and Flores, 2016; Ahmadov et al., 2021; Ikot et al., 2022). These additional terms can account for effects such as relativistic corrections, spin-dependent interactions, a medium of finite temperature, and non-perturbative QCD contributions. The generalized Cornell potential has been shown to improve the description of quarkonium bound states and the mass spectra (Ahmadov et al., 2021; Ikot et al., 2022).

Therefore, to the best of our knowledge, in the framework of non-relativistic quantum mechanics, there has been no published research work that has applied the Hellmann-Feynman Theorem for a quantum Hamiltonian whose potential term, a generalized Cornell potential, is not temperature-dependent to analytically obtain the expectation values for the dynamical variables: the square of the inverse of position, the inverse of position, the square of momentum, and the kinetic energy for their corresponding quantum states for charmonium and bottomium mesons in the framework of non-relativistic quantum mechanics. Hence, this research paper would be organized as follows: in Section 2 the Nikiforov–Uvarov (NU) method (Nikiforovo and Uvarov 1988) would be given a brief recapitulation; in Section 3 the NU method would be applied to the Schrödinger equation with the generalized Cornell potential model to obtain the bound state energy spectra equation, for which special cases would also be discussed; in Section 4 from the obtained bound state energy spectra, the mass spectra equation would be derived, for which through fitting with available experimental data the potential parameters would be defined for the charmonium and bottomnium mesons, respectively; in Section 5 the expectation value equations for the dynamical variables and for their corresponding quantum states for charmonium and bottomnium mesons would be carried out; in Section 6 the results applied to the charmonium and bottomnium systems would be discussed; and in Section 7 our conclusion would be presented.

Review of the Nikiforov-Uvarov (NU) method

The NU method is a mathematical invention proposed by Nikiforovo and Uvarov (1988) to transform hypergeometric-type second-order differential equations, such as the Jacobi differential equation, Laguerre differential equation, Bessel differential equation, Hermite differential equation, and any Schrödinger-like differential equation into a second-order differential equation via a coordinate transformation $x = x(r)$, of the form

$$g''(x) + \frac{\tilde{\tau}(x)}{\sigma(x)} g'(x) + \frac{\tilde{\sigma}(x)}{\sigma^2(x)} g(x). \tag{2.1}$$

Where $\tilde{\sigma}(x)$, and $\sigma^2(x)$ are polynomials of at most second degree and $\tilde{\tau}(x)$ is a first-degree polynomial. The exact solution to Eq. (2.1) can be found by the transformation

$$g(x) = \Phi(x) y_n(x). \tag{2.2}$$

This transformation reduces Eq. (2.1) into a hypergeometric-type equation of the form

$$\sigma''(x) y_n''(x) + \tau(x) y_n'(x) + \lambda y_n(x). \tag{2.3}$$

The function $\Phi(x)$ can be defined as the logarithm derivative

$$\frac{\Phi'(x)}{\Phi(x)} = \frac{\pi(x)}{\sigma(x)} \tag{2.4}$$

with $\pi(x)$ being at most a first-degree polynomial. The second part of $\Psi(x)$ which is $y_n(x)$ in Eq. (2.3), is the hypergeometric function with its polynomial solution given by the Rodrigues relation

$$y_n(x) = \frac{B_n}{\rho(x)} \frac{d^n}{dz^n} [\sigma^n(x) \rho(x)], \tag{2.5}$$

where B_n is the normalization constant and $\rho(x)$ is the weight function which must satisfy the condition

$$[\sigma(x)\rho(x)]' = \tau(x)\rho(x) \tag{2.6}$$

with

$$\tau(x) = \tilde{\tau}(x) + 2\pi(x) \tag{2.7}$$

It is customary that when employing the NU method, one should observe that the derivative $\tau(x)$ with respect to x should be negative. The eigenfunctions and eigenvalues can be derived using the definitions of the subsequent function $\pi(x)$ and parameter λ, respectively:

$$\pi(x) = \frac{\sigma'(x) - \tilde{\tau}(x)}{2} \pm \sqrt{\left(\frac{\sigma'(x) - \tilde{\tau}(x)}{2}\right)^2 - \tilde{\sigma}(x) + k\sigma'(x)} \tag{2.8}$$

and

$$\lambda = k + \pi(x) \tag{2.9}$$

The value of k can be obtained by setting the discriminant in Eq. (2.8) to zero. Therefore, the new eigenvalue equation can be written as

$$\lambda = \lambda_n = -n\tau'(x) - \frac{n(n-1)}{2} \sigma''(x) \tag{2.10}$$

3. Bound State Solution of the Radial Schrödinger Equation for the generalized Cornell Potential Model

The Schrödinger wave equation with the transformation $R(r) = \frac{U(r)}{r}$, can be written as (Ikot et al., 2022)

$$\frac{d^2 R_n(r)}{dr^2} + \frac{2\mu}{\hbar^2}\left[E - \frac{l(l+1)\hbar^2}{2\mu r^2} - V(r)\right] R_n(r) = 0. \tag{3.1}$$

Considering equation 3.1, the total wave function is written as

$$R(r) = \frac{U(r)}{r} \tag{3.2}$$

Let $V(r)$ be the Generalised Cornell Potential model, which is defined a

$$V(r) = ar^2 + br - \frac{c}{r} + \frac{d}{r^2} + e \tag{3.3}$$

where the 3rd and 4th term of Eq. 3.3 accounts for weak heavy quark-antiquark $(Q - \bar{Q})$ interaction at short distances for high energies and the 1st 2nd and 4th terms of Eq. 3.3 accounts for the strong heavy quark-antiquark $Q - \bar{Q}$ interaction at long distances for low energies

Substituting Eq. (3.3) into Eq. (3.1), and writing the radial part of the Schrödinger equation, we obtained

$$R''(r) + \frac{2}{r}R'(r) + \frac{2\mu}{\hbar^2}\left[E - \frac{l(l+1)\hbar^2}{2\mu r^2} - V(r)\right]R(r) = 0 \tag{3.4}$$

$$R''(r) + \frac{2}{r}R'(r) + \frac{2\mu}{\hbar^2}\left[E - \frac{l(l+1)\hbar^2}{2\mu r^2} - \left(ar^2 + br - \frac{c}{r} + \frac{d}{r^2} + e\right)\right]R(r) = 0 \tag{3.5}$$

Suppose we make the transformation $R(r) \to F(r)$ by setting $R(r) = \frac{1}{r}F(r)$, then

$$R'(r) = -\frac{1}{r^2}F(r) + \frac{1}{r}F'(r) \tag{3.6}$$

and

$$R''(r) = \frac{2F(r)}{r^3} - \frac{F(r)}{r^2} - \frac{F'(r)}{r^2} + \frac{1}{r}F''(r) \tag{3.7}$$

Substituting Eq. 3.6 and Eq. 3.7 into Eq. 3.5, we can write the radial equation as

$$F''(r) + \frac{2\mu}{\hbar^2}\left[E - \frac{l(l+1)\hbar^2}{2\mu r^2} - \left(ar^2 + br - \frac{c}{r} + \frac{d}{r^2} + e\right)\right]F(r) = 0 \tag{3.8}$$

were μ is the reduced mass for the quarkonium particle, for charmonium $\mu = \frac{m_c}{2}$ and bottomonium $\mu = \frac{m_b}{2}$.

The primary rationale for performing multiple transformations on the Schrödinger equation is to express the radial Schrödinger equation, as delineated in Eq. 3.5, in a hypergeometric format.

$$F''(r) + \frac{\tilde{\tau}(r)}{\sigma(r)}F'(r) + \frac{\tilde{\sigma}(r)}{\sigma^2(r)}F'(r). \tag{3.9}$$

which is actually of the same hypergeometric-type as stated in Eq. 3.. Hence, carrying out another transformation $F(r) \to g(x)$ by changing the variable $r = 1/x$ we have

$$\frac{d}{dr} = \frac{dx}{dr}\frac{d}{dx} = \left(\frac{-1}{r^2}\right)\frac{d}{dx} = -x^2\frac{d}{dx} \tag{3.10}$$

and

$$\frac{d^2}{dr^2} = \left(\frac{-1}{r^2}\right)\frac{d}{dx}\left(\frac{-1}{r^2}\right)\frac{d}{dx} = -x^2\left[-2x\frac{d}{dx} - x^2\frac{d^2}{dx^2}\right] \tag{3.11}$$

Substituting Eq. 3.10 and Eq. 3.11 into Eq. 3.5, we now obtain our desired hypergeometric-type equation, which we can write the radial Schrödinger equation as:

$$g''(x) + \frac{2x}{x^2}g'(r) + \frac{2\mu}{\hbar^2 x^4}\left[E - \frac{\hbar^2 l(l+1)x^2}{2\mu} - e - \frac{a}{x^2} - \frac{b}{x} + cx - dx^2\right]g(x) = 0 \tag{3.12}$$

To obtain an analytical solution to the Schrödinger equation as delineated in Eq. 3.12 utilizing the NU method, we will employ an approximation scheme akin to the Pekeris approximation method concerning the terms $\frac{a}{x^2}$ and $\frac{b}{x}$. Consequently, we shall perform a power series expansion around the minimum interaction distance r_o (or $\delta = 1/r_o$) for both terms in the x-space concerning the heavy quark and antiquark system. Consequently, the expansion for both terms will be executed by substituting $y = x - \delta$, then

$$\frac{b}{x} = \frac{b}{y+\delta} = \frac{b}{\delta}\left(1 + \frac{x-\delta}{\delta}\right)^{-1} \tag{3.13a}$$

$$\frac{b}{x} \approx \frac{b}{\delta}\left(1 - \frac{y}{\delta} + \frac{y^2}{\delta^2}\right) \tag{3.13b}$$

$$\frac{b}{x} \approx \frac{b}{\delta}\left(1 - \frac{x-\delta}{\delta} + \frac{(x-\delta)^2}{\delta^2}\right) \tag{3.13c}$$

$$\frac{b}{x} \approx \frac{b}{\delta}\left(\frac{\delta^3 - x\delta^2 + \delta^3 + x^2\delta + \delta^3 - 2x\delta^2}{\delta^3}\right) \tag{3.13d}$$

$$\frac{b}{x} \approx \frac{b}{\delta}\left(\frac{3\delta^3 - 3x\delta^2 + x^2\delta}{\delta^3}\right) \tag{3.13e}$$

$$\frac{b}{x} \approx b\left(\frac{3}{\delta} - \frac{3x}{\delta^2} + \frac{x^2\delta}{\delta^3}\right) \tag{3.13f}$$

and

$$\frac{a}{x^2} = \frac{a}{(y+\delta)^2} = \frac{a}{\delta^2}\left(1 + \frac{x-\delta}{\delta}\right)^{-2} \tag{3.13g}$$

$$\frac{a}{x^2} \approx \frac{a}{\delta^2}\left(1 - \frac{2y}{\delta} + \frac{3y^2}{\delta^2}\right) \tag{3.13h}$$

$$\frac{a}{x^2} \approx \frac{a}{\delta}\left(1 - \frac{2(x-\delta)}{\delta} + \frac{3(x-\delta)^2}{\delta^2}\right) \tag{3.13i}$$

$$\frac{a}{x^2} \approx \frac{a}{\delta^2}\left(\frac{\delta^3 - 2x\delta^2 + 2\delta^3 + 3x^2\delta + 3\delta^3 - 6x\delta^2}{\delta^3}\right) \tag{3.13j}$$

$$\frac{a}{x^2} \approx a\left(\frac{6}{\delta^2} - \frac{8x}{\delta^3} + \frac{3x^2}{\delta^4}\right) \tag{3.13k}$$

Substituting the approximate forms of the terms $\frac{a}{x^2}$ and $\frac{b}{x}$ from Eq. 3.13k and Eq. 3.13f respectively, into Eq. 3.12, we obtain

$$g''(x) + \frac{2x}{x^2}g'(x) + \frac{2\mu}{\hbar^2 x^4}\left[E - \frac{\hbar^2 l(l+1)x^2}{2\mu} - e - a\left(\frac{6}{\delta^2} - \frac{8x}{\delta^3} + \frac{3x^2}{\delta^4}\right) - b\left(\frac{3}{\delta} - \frac{3x}{\delta^2} + \frac{x^2\delta}{\delta^3}\right) + cx - dx^2\right]g(x) = 0 \tag{3.14}$$

To make Eq. (3.24) more compact we introduce the new variables

$$M = \frac{-2\mu}{\hbar^2}\left(E - e - \frac{3b}{\delta} - \frac{6a}{\delta^2}\right), \tag{3.15a}$$

$$N = \frac{2\mu}{\hbar^2}\left(\frac{3b}{\delta^2} + c - \frac{8a}{\delta^3}\right), \tag{3.15b}$$

and

$$O = \frac{2\mu}{\hbar^2}\left(-\frac{\hbar^2 l(l+1)}{2\mu} - \frac{b}{\delta^3} - d - \frac{3a}{\delta^4}\right) \tag{3.15c}$$

Substituting Eq. 3.15a, Eq. 3.15b and Eq. 3.15c into Eq. 3.14, we obtain

$$g''(x) + \frac{2x}{x^2}g'(x) + \frac{1}{x^4}[-M + Nx + Ox^2]g(x) = 0 \tag{3.16}$$

Hence, we can apply NU-method to define the bound the energy eigen value and eigenstate solution; by comparing Eq. 3.16 with Eq. 3.

$$\tilde{\tau}(x) = 2x,\ \sigma(x) = x^2,\ \tilde{\sigma}(x) = (-M + Nx + Ox^2) \tag{3.17}$$

where the function $\pi(x)$, the polynomial of degree at most one is defined as

$$\pi(x) = \pm\sqrt{(k - Q)x^2 - Nx + H} \tag{3.18}$$

after we had substituted Eq. 3.17 into Eq. 3.8 to define the polynomial function $\pi(x)$ of at most degree one for this problem.

The constant parameter in Equation 3.18 can be determined by utilizing the condition that the expression under the square root possesses a double root, indicating that its discriminant is zero. Consequently, we derive

$$k = \frac{1}{4M}(N^2 + 4MO) \tag{3.19}$$

Substituting Eq. 3.19 into Eq. 3.18 we obtain

$$\pi(x) = \pm\sqrt{\left(\frac{N^2+4MO}{4M} - O\right)x^2 - Nx + M} \tag{3.20}$$

$$\pi(x) = \pm\sqrt{\left(\frac{N^2}{4M}\right)x^2 - Nx + M} \tag{3.21}$$

$$\pi(x) = \pm\frac{1}{2\sqrt{M}}(Nx - 2M) \tag{3.22}$$

The polynomial function $\pi(x)$ presented in Eq. 3.22 has two forms, and according to the NU-method, a specific form of the polynomial function $\pi(x)$ is required for the function $\tau(x)$ defined in Eq. 3.7 to exhibit a negative derivative with respect to the independent variable x.

$$\pi(x) = -\frac{1}{2\sqrt{M}}(Nx - 2M), \tag{3.23}$$

Taking the derivative of the polynomial function $\pi(x)$ with respect to x-space, we obtain

$$\pi'(x) = -\frac{N}{2\sqrt{M}} \tag{3.24}$$

Substituting Eq. 3.17 and Eq. 3.23 into Eq. 3.7, we obtain

$$\tau(x) = 2x - \frac{Nx}{\sqrt{M}} + 2\sqrt{M} \tag{3.25}$$

Taking the derivative of the function $\tau(x)$ with respect to x-space, we obtain

$$\tau'(x) = 2 - \frac{N}{\sqrt{M}} \tag{3.26}$$

To obtain the energy eigen value, we would substitute Eq. 3.19, Eq. 3.22, Eq. 3.26 and Eq. 3.17 into Eq. 3.10, and we would get

$$\sqrt{M} = \frac{N}{[1 + 2n \pm \sqrt{1 - 4O}]} \tag{3.27}$$

If we recall Eq. 3.20, Eq. 3.21 and Eq. 3.22 for the variables N, M and O, the non-degenerate energy spectrum E_n in Eq. 3.27 can be written in the form:

$$E_{nl} = e + \frac{3b}{\delta} + \frac{6a}{\delta^2} - \frac{\hbar^2}{2\mu}\left[\frac{\frac{2\mu}{\hbar^2}\left(\frac{3b}{\delta^2} + c + \frac{8a}{\delta^3}\right)}{(1+2n) \pm \sqrt{1 + 4l(l+1) + \frac{8\mu b}{\hbar^2\delta^3} + \frac{8\mu d}{\hbar^2} + \frac{24a\mu}{\hbar^2\delta^4}}}\right]^2 \tag{3.28}$$

4. Calculation of the Mass Spectra of the Heavy Quarkonium System

To compute the mass spectra M for the heavy quarkonium system, including charmonium and bottomonium mesons, we will utilize the relation provided by Quigg and Rosner (1979), Vega and Flores (2016), Soni et al. (2018), and Ahmadov et al. (2019, 2021).

$$M = m_q + m_{\bar{q}} + E_{nl} \tag{4.1}$$

Where m_q is the mass of the heavy quark, $m_{\bar{q}}$ the mass of the heavy anti-quark, and E_{nl} the non-relativistic energy eigenvalue for our generalized Cornell potential model $V(r)$. Consequently, given that the charmonium state of spin 1 was initially documented by Aubert et al. (1974) and Augustin et al. (1974), and the bottomonium ground state was reported by Herb et al. (1977), their findings indicate that charmonium and bottomonium particles and antiparticles possess nearly identical masses; thus, the mass of the heavy quark and heavy antiquark can be mathematically approximated as $m_q \approx m_{\bar{q}}$. Consequently, Eq. 4.1 can be reformulated as

$$M = 2m_q + E_{nl} \tag{4.2}$$

By substituting the energy spectrum delineated in Eq. 3.28 into our reformulated mass spectral distribution equation, we derived the mass spectral distribution formula for the quarkonium system under the Generalized Cornell Potential model as

$$M = 2m_q + e + \frac{3b}{\delta} + \frac{6a}{\delta^2} - \frac{\hbar^2}{2\mu}\left[\frac{\frac{2\mu}{\hbar^2}\left(\frac{3b}{\delta^2}+c+\frac{8a}{\delta^3}\right)}{(1+2n)+\sqrt{1+4l(l+1)+\frac{8\mu b}{\hbar^2\delta^3}+\frac{8\mu d}{\hbar^2}+\frac{24a\mu}{\hbar^2\delta^4}}}\right]^2 \tag{4.3}$$

5. Expectation Value Equations for the Dynamical variables: $\langle r^{-2}\rangle$, $\langle r^{-1}\rangle$, $\langle T\rangle$, and $\langle p^2\rangle$ via the Hellmann-Feynman theorem

The Hellmann-Feynman theorem for a non-degenerate bound state quantum system which states that:

$$\frac{\partial E_{nl}(\lambda)}{\partial\lambda} = \left\langle \psi_{nl}(\lambda)\middle|\frac{\partial \hat{H}_\lambda}{\partial\lambda}\middle|\psi_{nl}(\lambda)\right\rangle \tag{5.1}$$

where $\hat{H}_\lambda$ is the Hamiltonian Hermitian operator that depends on the parameter λ, $|\psi_{nl}(\lambda)\rangle$ is the eigenstate of the Hamiltonian that depends implicitly on λ and E_{nl} is the energy eigenvalue of the bound steady-state system.
The effective Hamiltonian of the Generalized Cornell Potential model $V(r)$ is given by:

$$\hat{H} = -\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + \frac{\hbar^2}{2\mu}\frac{l(l+1)}{r^2} + ar^2 + br - \frac{c}{r} + \frac{d}{r^2} + e \tag{5.2}$$

Consequently, with our defined energy eigenvalue equation of Eq. 3.28, we would derive the mathematical expressions for the expectation values for the square of the inverse of position $\langle r^{-2}\rangle$,, the inverse of position, $\langle r^{-1}\rangle$, the square of momentum, $\langle p^2\rangle$and the kinetic energy $\langle T\rangle$ corresponding to s, p, d, and f states for the heavy quarkonium (charmonium and bottomonium).

5.1 Expectation value equation for the inverse of position $\langle r^{-1}\rangle$

By computing the partial derivative of the effective Hamiltonian of the Generalized Cornell Potential model, as delineated in Eq. 5.2, with respect to the potential parameter c, we obtain the following:

$$\frac{\partial\hat{H}}{\partial c} = \frac{-1}{r} \tag{5.3}$$

Taking the partial derivative of the non-degenerate energy eigenvalue equation for the generalized Cornell Potential model as expressed as Eq. 3.28 with respect to the potential parameter c, we get:

$$\frac{\partial E_{nl}}{\partial c} = -\frac{\frac{2\mu}{\hbar^2}\left(\frac{6b}{\delta^2}+2c+\frac{16a}{\delta^3}\right)}{\left[(1+2n)+\sqrt{1+4l(l+1)+\frac{8\mu b}{\hbar^2\delta^3}+\frac{8\mu d}{\hbar^2}+\frac{24a\mu}{\hbar^2\delta^4}}\right]^2} \tag{5.4}$$

Applying the Hellmann-Feynman theorem as articulated in Eq. 5.1, by designating the parameter λ as the Generalized Cornell Potential parameter and incorporating Eq. 5.3 and Eq. 5.4 into Eq. 5.1, we derive the expectation value equation for the inverse of position.
$\langle r^{-1}\rangle$ as:

$$\langle r^{-1}\rangle = \frac{\frac{2\mu}{\hbar^2}\left(\frac{6b}{\delta^2}+2c+\frac{16a}{\delta^3}\right)}{\left[(1+2n)+\sqrt{1+4l(l+1)+\frac{8\mu b}{\hbar^2\delta^3}+\frac{8\mu d}{\hbar^2}+\frac{24a\mu}{\hbar^2\delta^4}}\right]^2} \quad (5.5)$$

5.2 Expectation value equation for the square of inverse of position $\langle r^{-2}\rangle$

Calculating the partial derivative of the effective Hamiltonian of the Generalized Cornell Potential model, as delineated in Eq. 5.2, with respect to the angular momentum quantum number l yields the following:

$$\frac{\partial \hat{H}}{\partial l} = \frac{\hbar^2}{2\mu r^2}(2l+1) \quad (5.6)$$

Taking the partial derivative of the non-degenerate energy eigenvalue equation for the Generalised Cornell Potential model as expressed as Eq. 3.28 with respect to the angular momentum quantum number l, we get:

$$\frac{\partial E_{nl}}{\partial l} = \frac{4\mu(4l+2)\left(\frac{3b}{\delta^2}+c+\frac{8a}{\delta^3}\right)^2\left(\frac{1}{\hbar^2}\right)}{\sqrt{1+4l(l+1)+\frac{8\mu b}{\hbar^2\delta^3}+\frac{8\mu d}{\hbar^2}+\frac{24a\mu}{\hbar^2\delta^4}}\left((1+2n)+\sqrt{1+4l(l+1)+\frac{8\mu b}{\hbar^2\delta^3}+\frac{8\mu d}{\hbar^2}+\frac{24a\mu}{\hbar^2\delta^4}}\right)^3} \quad (5.7)$$

By applying the Hellmann-Feynman theorem as articulated in Eq. 5.1, designating the parameter λ as the angular momentum quantum number l, and substituting Eq. 5.6 and Eq. 5.7 into Eq. 5.1, we derive the expectation value equation for the square inverse of position. $\langle r^{-2}\rangle$ as:

$$\langle r^{-2}\rangle = \frac{8\mu\left(\frac{3b}{\delta^2}+c+\frac{8a}{\delta^3}\right)^2\left(\frac{1}{\hbar^4}\right)}{\sqrt{1+4l(l+1)+\frac{8\mu b}{\hbar^2\delta^3}+\frac{8\mu d}{\hbar^2}+\frac{24a\mu}{\hbar^2\delta^4}}\left((1+2n)+\sqrt{1+4l(l+1)+\frac{8\mu b}{\hbar^2\delta^3}+\frac{8\mu d}{\hbar^2}+\frac{24a\mu}{\hbar^2\delta^4}}\right)^3} \quad (5.8)$$

5.3 Expectation value equation for the kinetic energy $\langle T\rangle$

Taking the partial derivative of the effective Hamiltonian of the Generalised Cornell Potential model as expressed as Eq. 5.2 with respect to the reduced mass of the heavy quarkonium μ, we would get:

$$\frac{\partial \hat{H}}{\partial \mu} = \frac{-1}{\mu}\left(-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2}+\frac{\hbar^2}{2\mu}\frac{l(l+1)}{r^2}\right) \quad (5.9)$$

Taking the partial derivative of the non-degenerate energy eigenvalue equation for the Generalised Cornell Potential model as expressed as Eq. 3.28 with respect to the reduced mass of the heavy quarkonium μ, we would get:

$$\frac{\partial E_{nl}}{\partial \mu} = \frac{\frac{4\mu}{\hbar^2}\left(\frac{4b}{\hbar^2\delta^3}+\frac{4d}{\hbar^2}\right)\left(\frac{3b}{\delta^2}+c+\frac{8a}{\delta^3}\right)^2}{\sqrt{1+4l(l+1)+\frac{8\mu b}{\hbar^2\delta^3}+\frac{8\mu d}{\hbar^2}+\frac{24a\mu}{\hbar^2\delta^4}}\left((1+2n)+\sqrt{1+4l(l+1)+\frac{8\mu b}{\hbar^2\delta^3}+\frac{8\mu d}{\hbar^2}+\frac{24a\mu}{\hbar^2\delta^4}}\right)^3} - \frac{\frac{2}{\hbar^2}\left(\frac{3b}{\delta^2}+c+\frac{8a}{\delta^3}\right)^2}{\left((1+2n)+\sqrt{1+4l(l+1)+\frac{8\mu b}{\hbar^2\delta^3}+\frac{8\mu d}{\hbar^2}+\frac{24a\mu}{\hbar^2\delta^4}}\right)^2} \quad (5.10)$$

Using the Hellmann-Feynman theorem as mathematically expressed in Eq. 5.1, when we set the parameter λ to be the reduced mass of the heavy quarkonium μ; substituting Eq. 5.10 and Eq. 5.11 into Eq. 5.1, we obtain the expectation value equation for the kinetic energy $\langle T\rangle$ as:

$$\langle T\rangle = -\frac{\frac{4\mu^2}{\hbar^2}\left(\frac{4b}{\hbar^2\delta^3}+\frac{4d}{\hbar^2}\right)\left(\frac{3b}{\delta^2}+c+\frac{8a}{\delta^3}\right)^2}{\sqrt{1+4l(l+1)+\frac{8\mu b}{\hbar^2\delta^3}+\frac{8\mu d}{\hbar^2}+\frac{24a\mu}{\hbar^2\delta^4}}\left((1+2n)+\sqrt{1+4l(l+1)+\frac{8\mu b}{\hbar^2\delta^3}+\frac{8\mu d}{\hbar^2}+\frac{24a\mu}{\hbar^2\delta^4}}\right)^3} + \frac{\frac{2\mu}{\hbar^2}\left(\frac{3b}{\delta^2}+c+\frac{8a}{\delta^3}\right)^2}{\left((1+2n)+\sqrt{1+4l(l+1)+\frac{8\mu b}{\hbar^2\delta^3}+\frac{8\mu d}{\hbar^2}+\frac{24a\mu}{\hbar^2\delta^4}}\right)^2} \tag{5.11}$$

5.4 Expectation value equation for square of momentum $\langle p^2\rangle$

When we recalled from our memory the relation between kinetic energy T and momentum p for a particle or a wave, it was given as;

$$T = \frac{p^2}{2\mu} \tag{5.12}$$

If we set the parameter λ in Eq. 5.1 to be the reduced mass of the heavy quarkonium μ, the expectation value equation for the kinetic energy $\langle T\rangle$ is given as;

$$-\frac{\langle T\rangle}{\mu} = \left\langle \psi(\mu)\left|\frac{\partial \hat{H}(\mu)}{\partial\mu}\right|\psi(\mu)\right\rangle \tag{5.13}$$

Hence, the relation between the expectation value equation for the kinetic energy $\langle T\rangle$ and the expectation value equation for the square of the momentum p for a particle or a wave can now be mathematically expressed as;

$$-\frac{\langle T\rangle}{\mu} = -\frac{\langle p^2\rangle}{2\mu^2} \tag{5.14}$$

Therefore, it is now clear that after applying the Hellmann-Feynman theorem as mathematically expressed in Eq. 5.1, when we set the parameter λ to be the reduced mass of the heavy quarkonium μ, the expectation value equation for square of momentum $\langle p^2\rangle$ is given as;

$$\langle p^2\rangle = -\frac{\frac{8\mu^3}{\hbar^2}\left(\frac{4b}{\hbar^2\delta^3}+\frac{4d}{\hbar^2}\right)\left(\frac{3b}{\delta^2}+c+\frac{8a}{\delta^3}\right)^2}{\sqrt{1+4l(l+1)+\frac{8\mu b}{\hbar^2\delta^3}+\frac{8\mu d}{\hbar^2}+\frac{24a\mu}{\hbar^2\delta^4}}\left((1+2n)+\sqrt{1+4l(l+1)+\frac{8\mu b}{\hbar^2\delta^3}+\frac{8\mu d}{\hbar^2}+\frac{24a\mu}{\hbar^2\delta^4}}\right)^3} + \frac{\frac{4\mu^2}{\hbar^2}\left(\frac{3b}{\delta^2}+c+\frac{8a}{\delta^3}\right)^2}{\left((1+2n)+\sqrt{1+4l(l+1)+\frac{8\mu b}{\hbar^2\delta^3}+\frac{8\mu d}{\hbar^2}+\frac{24a\mu}{\hbar^2\delta^4}}\right)^2} \tag{5.15}$$

6. Results and Discussion

The study focused on the non-relativistic limit of quantum theory, employing the Schrödinger wave equation with the generalized Cornell potential model to derive the bound-state energy spectra and mass spectra equations. The results were utilized for charmonium and bottomonium mesons to provide phenomenological predictions for their mass spectra, as well as to derive the expectation values of the dynamical variables ($\langle r^{-2}\rangle$, $\langle r^{-1}\rangle$, $\langle T\rangle$, and $\langle p^2\rangle$) for the quantum states of charmonium and bottomonium.

In Fig. 1, we present the potential of the generalized Cornell potential model, $V(r)$ as a function of inter-quark distance $r(GeV^{-1})$ for charmonium $(c\bar{c})$ and bottomonuim $(b\bar{b})$ mesons. This is based on fitting the derived mass spectra equation from Eq. 4.3, which allowed us to define the potential parameters (a, b, c, d, and e) of the generalized Cornell potential. The results indicate that as the inter-quark distance decreases, the potential energy increases monotonically. This suggests that the bottomonium meson possesses a higher bound state energy than the charmonium meson.

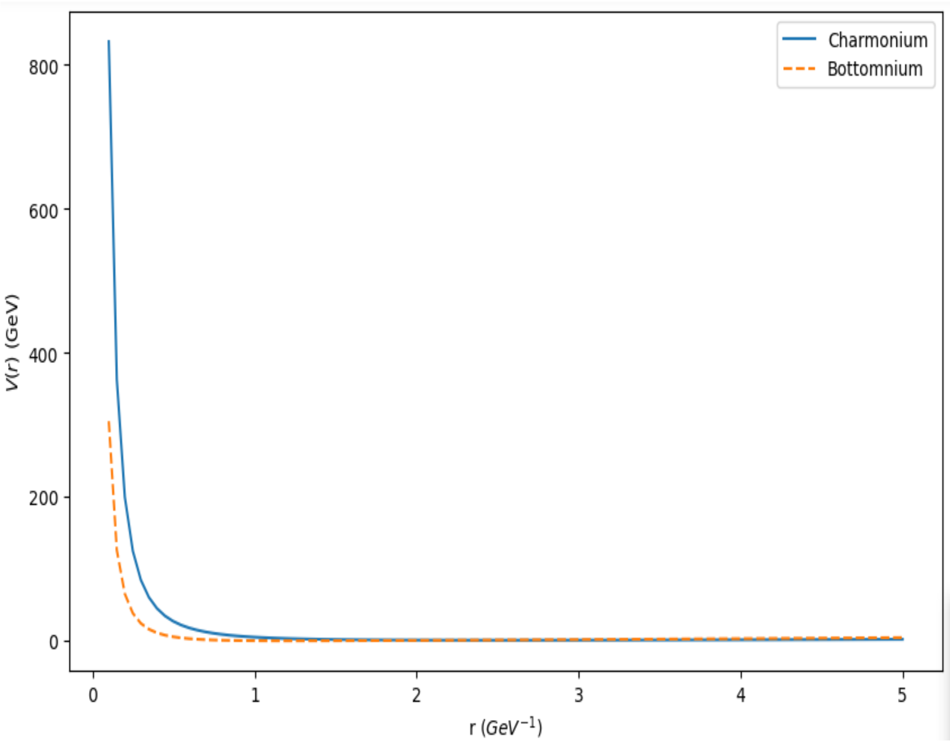

Fig. 1: Generalized Cornell potential model $V(r)$ as a function of inter-quark distance $r(GeV^{-1})$ for heavy quarkonium (Charmonium and Bottomonium).

In Tables 1 and 2, we present our phenomenological predictions of the mass spectra for charmonium and bottomonium, expressed in units GeV of their numerical mass values $m_c = 1.209\ GeV$ and $m_b = 4.823\ GeV$. Their corresponding reduced masses are also provided (Beringer et al., 2012). To determine the values of the potential parameters a, b, c, and d, we fitted the derived mass spectra equation, as stated in Eq. 4.3, to experimental data compiled by Beringer et al. (2012). As indicated in Tables 1 and 2, we found that our phenomenological predictions of the mass spectra for charmonium and bottomonium across various s, p, and d states are consistent with the currently available experimental data, as well as with theoretical predictions made by other researchers using different analytical techniques. The following relation is utilized to compute the absolute percentage deviation (σ) of our phenomenological predictions from the experimental data (Ikot et al., 2022).

$$\sigma = \frac{100}{N} \sum \left| \frac{T_p - T_{ED}}{T_{ED}} \right| \tag{6.1}$$

where T_{ED} is the experimental data for $(c\bar{c})$ and $(b\bar{b})$ the mesons, The symbol T_p represents the phenomenologically predicted results, while N indicates the number of available experimental data points. Furthermore, the percentage error (ρ) was calculated using the equation provided by Ikot et al. (2022).

$$\rho = \frac{\sum(\text{ARD})}{\sum T_{ED}} \times 100\% \tag{6.2}$$

The absolute percentage deviation (σ) of our phenomenological expected findings from the experimental data is 1.09 for charmonium mesons and 0.46 for bottomonium mesons, respectively. The percentage error (ρ) between our projected outcomes and the experimental data for charmonium and bottomonium is 0.99% and 0.44%, respectively.

After applying the Hellmann-Feynman Theorem, we obtained results for the expectation values of the dynamical variables $\langle r^{-2}\rangle$, $\langle r^{-1}\rangle$ $\langle T\rangle$, and $\langle p^2\rangle$ for the quarkonium states, specifically charmonium and bottomonium, as presented in Tables 3 and 4.

Table 1: The phenomenological prediction of the mass spectrum for Charmonium GeV compared to other research data, with a value of 1.209 GeV with fitted parameters;a = 0.074581 GeV³, b = 0.0139996 GeV², c = 4.450992, d = 8.768755 GeV⁻¹, e = 0.065473 GeV, and δ = 0.431597 GeV.

Table 2: The phenomenological predicted mass spectrum for bottomonium GeV compared to other research data

State	Present work	Ahmadov et al (2019)	Ahmadov et al (2020)	Faustov et al (1997)	Kumar and Chand(2012)	Al-Jamel and Widyan (2012)	Experiment (Beringer et al, 2012)
1s	3.096670801	3.0969	3.098	3.068	3.078	3.0969	3.096
2s	3.68714876	3.686	3.687	3.697	3.581	3.686	3.686
3s	4.038249161	4.04143	4.042	4.144	4.085	3.984	4.04
4s	4.263831278	4.27086	4.272		4.589	4.15	4.263
5s	4.417306234	4.42783	4.429				4.421
1p	3.260108148	3.25581	3.256	3.526	3.415	3.433	
2p	3.781270627	3.77951	3.78	3.993	3.917	3.91	3.773
3p	4.097317594	4.09997	4.1				
4p	4.303301362	4.31021	4.31				
5p	4.444972562	4.45555	4.456				
1d	3.513093893	3.50471	3.505	3.829	3.749	3.767	

at 4.823 GeV with fitted parameters ;a = 0.1374 GeV³, b = 0.064869505 GeV², c = 5.2105, d = 3.558096 GeV⁻¹, e = 0.982250689 GeV, and δ = 1.3165 GeV.

State	Present work	Ahmadov et al (2019)	Ahmadov et al (2020)	Faustov et al (1997)	Kumar and Chand(2012)	Al-Jamel and Widyan (2012)	Experiment (Beringer et al, 2012)
1s	9.455196774	9.46	9.46	9.447	9.51	9.46	9.46
2s	10.02529873	10.023	10.023	10.012	10.038	10.023	10.023
3s	10.36155231	10.355	10.355	10.353	10.566	10.28	10.355
4s	10.57634074	10.5661	10.567	10.629	11.094	10.42	10.58
5s	10.72183104	10.7098					
1p	9.616397454	9.61781	9.619	9.9	9.862	9.84	10.26
2p	10.11731704	10.1127	10.114	10.26	10.39	10.16	
3p	10.41893992	10.4106	10.411				
4p	10.61450866	10.6038	10.604				
5p	10.74848774	10.7362	10.736				
1d	9.863709343	10.2567	9.863	10.155	10.214	10.14	10.164

Table 3: Phenomenologically predicted mass spectrum for Charmonium GeV, along with the calculated observables. These observables include the square of the inverse of position $\langle r^{-2}\rangle$, the inverse of position $\langle r^{-1}\rangle$, the square of momentum $\langle p^2\rangle$, and the kinetic energy corresponding to each state, derived using the Hellmann-Feynman Theorem.

State	mass spectrum (GeV)	$\langle r^{-1}\rangle$ (GeV^{-1})	$\langle r^{-2}\rangle$ (GeV^{-2})	$\langle T\rangle$ (GeV)	$\langle p^2\rangle$ (GeV^2)	Experiment (Beringer et al, 2012)
1s	3.096670801	0.311855879	0.0897038	0.916510855	1.108061624	3.096
2s	3.68714876	0.21423841	0.051077066	0.743663236	0.899088852	3.686
3s	4.038249161	0.156194698	0.031796504	0.60102332	0.726637194	4.04
4s	4.263831278	0.118901595	0.021118422	0.490892098	0.593488547	4.263

5s	4.417306234	0.093529205	0.014733326	0.406452479	0.491401047	4.421
1p	3.260108148	0.284836513	0.074453418	0.917959884	1.109813499	
2p	3.781270627	0.198678238	0.043372714	0.732840439	0.886004091	3.773
3p	4.097317594	0.146429539	0.027443165	0.589022972	0.712128773	
4p	4.303301362	0.112376424	0.018450322	0.480269371	0.580645670	
5p	4.444972562	0.088955424	0.012994208	0.397589414	0.480685602	
1d	3.513093893	0.243013058	0.05374504	0.888872093	1.074646361	

Table 4: The phenomenological predicted mass spectrum for bottomonium in GeV and the calculated observables: square of inverse of position $\langle r^{-2}\rangle$, the inverse of position $\langle r^{-1}\rangle$, square of momentum $\langle p^{2}\rangle$, and the kinetic energy $\langle T\rangle$ corresponding to each state by applying the Hellmann-Feynman Theorem.

State	mass spectrum (GeV)	$\langle r^{-1}\rangle$ (GeV^{-1})	$\langle r^{-2}\rangle$ (GeV^{-2})	$\langle T\rangle$ (GeV)	$\langle p^{2}\rangle$ (GeV^{2})	Experiment (Beringer et al, 2012)
1s	9.455196774	0.615907999	0.087868634	0.267610186	1.290683926	9.46
2s	10.02529873	0.420838352	0.049628720	0.362927793	1.750400743	10.023
3s	10.36155231	0.305653412	0.030718781	0.355727944	1.715675873	10.355
4s	10.57634074	0.232016960	0.020316070	0.321957544	1.552801234	10.58
5s	10.72183104	0.182107566	0.014127070	0.284162041	1.370513522	
1p	9.616397454	0.560914021	0.072489659	0.374024161	1.803918530	10.26
2p	10.11731704	0.389407319	0.041931247	0.404855168	1.952616474	9.46
3p	10.41893992	0.286034161	0.026397237	0.373540392	1.801585309	
4p	10.61450866	0.218959880	0.017679844	0.329663017	1.589964729	

5p	10.74848774	0.172983818	0.012414803	0.287300723	1.385651386
1d	9.863709343	0.476437021	0.051847183	0.485916844	2.343576939

As a result, the expectation values of the dynamical variables $\langle r^{-2}\rangle$and $\langle r^{-1}\rangle$ related to both charmonium and bottomonium mesons decreased with each increase in the principal quantum number across all quantum states $l = 0, 1, 2$ while the bottomonium system exhibited higher expectation values than the charmonium mesons.

Furthermore, regarding the expectation values of the dynamical variables $\langle T\rangle$, and $\langle p^2\rangle$,for charmonium and bottomonium mesons, each quantum state $l = 0, 1, 2$corresponding to these mesons showed a decrease in expectation values with every increase in the principal quantum number. While the bottomonium mesons had higher expectation values than the charmonium mesons for one dynamical variable $\langle p^2\rangle$, they lagged behind the charmonium mesons for another dynamical variable $\langle T\rangle$.

7. Conclusion

In the context of non-relativistic quantum theory via the Schrödinger wave equation, with the generalized Cornell potential model we have applied our results to make phenomenological prediction for the mass spectra distribution for the respective charmonium and bottomonium quantum states, and in applying Hellmann-Feynman Theorem, we have been able to analytically obtain the expectation values for the dynamical variables, $\langle \boldsymbol{r^{-2}}\rangle$, $\langle \boldsymbol{r^{-1}}\rangle$ $\langle \boldsymbol{T}\rangle$, and $\langle \boldsymbol{p^2}\rangle$. The said theorem has proven highly valuable tool for deriving expectation values analytically, avoiding sophisticated wavefunction integrations while providing physically meaningful observables. Expectation values obtained successfully describe structural and dynamical properties of quarkonium systems, especially charmonium and bottomonium.

Author Contributions:

Conceptualization, E.P. and E.P.; Methodology, E.P and E.E.; Software, U.A.U; Validation, E.P., E.E., and U.A.U.; Formal Analysis, E.P.; Writing—Original Draft, E.P.; Writing—Review & Editing, E.E. and U.A.U.; Visualization, E.P.; Supervision, E.E.

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